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# Time dependent Ginzburg–Landau model in the absence of translational invariance. Non-conserved order parameter domain growth

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**Abstract.** We have determined the static and dynamical properties of the Ginzburg–Landau model, with global coupling of the spherical type, on some non-translationally invariant lattices. Our solutions show that, in agreement with general theorems, fractal lattices with finite ramification do not display a finite temperature phase transition for any embedding dimension, $d$. On the other hand, the dynamical behaviour associated with the phase ordering dynamics of a non-conserved order parameter is non-trivial. Our analysis reveals that the domain size $R$ grows in time as $R(t) \sim t^z$ and relates this exponent to the three exponents which characterize the static and dynamical properties of fractal structures, namely the fractal dimension of the lattice $d_f$, the random walk dimension $d_w$ and the spectral dimension $d_s$. We also present a brief renormalization group treatment of the model. Finally, we have considered lattices with infinite ramification numbers which have spectral dimensions larger that 2 and show a finite temperature phase transition.

## 0. Introduction

In last few years considerable attention has been devoted to the understanding of growth phenomena in systems described by an order parameter $\phi(\boldsymbol{r}, t)$ [1, 2]. An important class of models introduced with the aim of studying the approach to equilibrium after a quench from a high temperature state is represented by the so-called time-dependent Ginzburg–Landau (GL) equation. Within such a phenomenological approach the order parameter evolves due to the presence of a deterministic force, as to minimize the free energy of the system, plus a stochastic contribution which takes into account the presence of thermal fluctuations. The free energy to be extremized contains a local part, which is, in general, a nonlinear function of the order parameter supplemented by a non-local term. Various forms have been considered in the case of continuum models for the non-local term, namely square gradient type $\frac{1}{2}\int \mathrm{d}^d r(\nabla\phi)^2$ interactions, long range type $\frac{1}{2}\int \mathrm{d}^d r \int \mathrm{d}^d r' \phi(r) w(r, r') \phi(r')$, (where $w(r, r')$ is a two body potential) together with their discrete counterparts that one can define on standard Euclidean lattices. For simple isotropic systems a fairly good understanding has been obtained: it has been realized that in the late stage the growth process depends on a single scaling length $L(t)$, the average domain size [2]. The scaling hypothesis also

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predicts that in the late stage $L \approx t^z$, where $z$ is a scaling exponent and that the structure factor $S(k,t) = \langle \phi(\boldsymbol{k},t)\phi(-\boldsymbol{k},t)\rangle$ displays the following scaling form

$$S(k,t) = t^{d\ z} f(kt^z)$$

where $d$ is the dimensionality of the embedding space, $k$ is the wavevector, and $f(y)$ is the scaling function. Moreover, the dynamical exponent, $z$, is rather universal and takes the values $\frac{1}{2}$ for a non-conserved order parameter, and $z = \frac{1}{3}$ if the order parameter is conserved†.

Little attention has been payed so far to fractal lattices where translational invariance is absent but another kind of property, the self-similarity is at work. Since the spectrum and the density of states on fractal lattices are extremely peculiar one should expect to observe unusual growth properties in these systems. Well known examples of fractal lattices, which have been thoroughly studied in the past, are the Sierpinski gaskets (SG). As remarked by Rammal and Toulouse, in self-similar structures the translational symmetry characterizing the Euclidean lattices is replaced by the dilation symmetry. As a consequence of such a weaker symmetry fractal lattices must be characterized by at least three dimensions instead of a single one. These dimensions are the fractal Hausdorff dimension $d_f$, the spectral dimension $d_s$ and the dimension $d$ of the embedding Euclidean space, on which $d_f$ and $d_s$ depend.

Equilibrium properties of spin models defined on SG were studied in the 1980s by several authors [3, 4] by means of real space renormalization group (RG) techniques. According to these studies at finite temperatures no phase transition can occur due to the finite order of ramification, $\mathcal{R}$, of these lattices. A finite order of ramification reflects the fact that, in these types of lattices, upon eliminating a finite number of lattice bonds one can isolate an arbitrarily large compact subset of the infinite system [3]. Some results concerning dynamical properties relative to SG have been obtained by means of coupled maps by Cosenza and Kapral [5] and Giacometti *et al* [6] within a RG study of the Langevin dynamics. In the present paper we shall investigate the behaviour of the spherical model on the $d = 2$ and $d = 3$ SG by means of an exact solution. In the present paper we extend our previous study of the dynamical properties of the GL model on fractal structures [7] and provide a detailed description of the relaxation process.

The paper is organized as follows: in section 1 we introduce the model, in section 2 we discuss the fractal lattice and in section 3 we derive the governing equations. In section 4 we analyse the equilibrium properties of the model and in section 5 the zero temperature relaxation, while in section 6 we include the effect of a finite temperature quench. In section 7 we conclude the analysis of SG with a brief renormalization group derivation of the scaling behaviour; in section 8 we discuss a self-affine lattice having spectral dimension larger than 2 and thus displaying a true thermodynamic phase transition. Finally, in the appendix we derive the equilibrium properties of the model by calculating the partition function.

## 1. The model

The two-dimensional SG is a self-similar object constructed by taking an equilateral triangle (the initiator) of linear size $L$. The first generation is obtained by halving each side and connecting the midpoints to form four equilateral triangles of equal shape and side $L/2$

† For scalar systems subject to conserved and non-conserved dynamics (models A and B) on discrete structures, however, there is some evidence that lattice anisotropies can determine non-universal behaviour and correlation functions can display non-universal features.

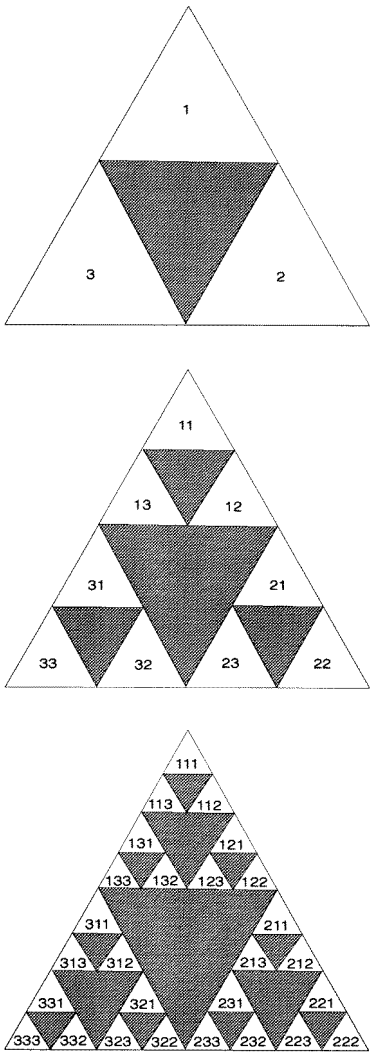


**Figure 1.** First three generations of SG for $d = 2$. The labels indicate the cell coding scheme.

and discarding the central one. One then repeats the same operation on each of the three remaining triangles and so forth up to the generation $n$, and eventually lets $n$ tend to infinity (see figure 1). Such a cascade process, in the case $d = 2$, after $n$ steps generates $N = 3^n$ equilateral triangles whose side is smaller than the side of the initiator by a factor $2^{-n}$. The fractal dimension of the gasket is $d_f = \ln(3)/\ln(2)$. For higher-dimensional embedding spaces one generalizes the above construction starting with a hyper-tetrahedron and after $n$ steps generates $N = (d + 1)^n$ hypertetrahedra and correspondingly $d_f = \ln(d + 1)/\ln(2)$ [8].

In order to provide a description of a phase ordering process on a fractal substrate we consider the well known GL model in which we take the local term to be a double well, whereas the non-local term, representing the energy cost required to create an

inhomogeneity, is proportional to

$$\tfrac{1}{2}\sum_i \sum_{j\in\mathcal{N}_i}[\phi_j - \phi_i]^2 = -\tfrac{1}{2}\sum_{i,j}\phi_i \Delta_{ij}\phi_j \tag{1}$$

where $i$ and $j$ are labels of the $(d+1)^n$ cells and the set $\mathcal{N}_i$ consists of the neighbours of the cell $i$.

The adjacency matrix $\mathbf{\Delta}$ can be viewed as the discrete version of the Laplacian operator on SG; in fact, one can write $\Delta_{ij}$ as a difference operator in analogy with the discrete representation of the Laplacian on Euclidean lattices.

An explicit expression of $\mathbf{\Delta}$ is obtained by defining a suitable labelling of the lattice sites in order to take advantage of the hierarchical structure, as we shall see below.

With these premises, the model is described by the following GL Hamiltonian:

$$H[\{\phi_i\}] = -\frac{1}{2}\sum_{i,j}\phi_i \Delta_{ij}\phi_j + \frac{r}{2}\sum_i^N {\phi_i}^2 + \frac{g}{4N}\left(\sum_i^N {\phi_i}^2\right)^2. \tag{2}$$

Notice that following the work of Cosenza and Kapral [5] we associate the field with the sites (the hypertetrahedra) and not with their vertices. With such a convention each site has $(d+1)$ neighbours, a part from those located at the $(d+1)$ vertices of the whole structure, which have only $d$ neighbours. The parameters $r$ and $g$ (with $r<0$ and $g>0$) represent the standard quadratic and quartic couplings of the GL model. The quartic term featuring in equation (2) is assumed to be of the form introduced by Berlin and Kac [9] because it allows a closed solution of the problem.

## 2. The lattice

Let us consider a SG embedded in $d=2$. A convenient labelling of the lattice sites, in this case, employs a ternary coding scheme as proposed in [5]. One establishes a correspondence between the cells belonging to the $n$th generation and the elements of a sequence, of length $n$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_i$ can take on one of the three values $1, 2, 3$. Such a labelling scheme is very convenient for the construction of the adjacency matrix $\Delta_{ij}$ on the SG and of the Cartesian coordinates of the cells (see figure 1).

At each stage of the construction of the fractal, the three newborn cells from the same parent cell, differ among themselves by the last element of the string $\boldsymbol{\alpha}$.

Thus, at the $n$th generation each cell is identified by a string of $n$ of such integers $\boldsymbol{\alpha}$. Two cells associated to strings $\boldsymbol{\alpha}$'s having the first $(n-1)$ elements identical derive from the same parent cell and are clearly nearest neighbours. Every cell also has a nearest-neighbour cell belonging to a different parent triangle (with the exception of the three cells located at the vertices of the whole structure). It can be easily verified that if a cell is identified by the sequence of the type $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{n-s-1}, \alpha_{n-s}, \alpha^s_{n-s+1})$, where $\alpha^s_k$ stands for $\underbrace{\alpha_k, \alpha_k, \ldots, \alpha_k}_{s}$ (with $1 \leqslant s \leqslant n-1$), its nearest-neighbour cell from a different parent cell is represented by the sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{n-s-1}, \alpha_{n-s+1}, \alpha^s_{n-s})$, which is obtained by the replacements $\alpha_{n-s} \to \alpha_{n-s+1}$ and $\alpha_{n-s+1} \to \alpha_{n-s}$. Finally, the three cells located at the vertices of the whole fractal, identified by a sequence of the type $\boldsymbol{\alpha} = (\alpha^n_1)$, have only two nearest neighbours, both belonging to the same parent cell.

Once each cell has been given the proper label, an adjacency $3^n \times 3^n$ matrix can be constructed by associating to each cell a natural number $j$, which is represented, for finite $n$, by the sequence $\boldsymbol{\alpha}$ as $j = 3^{n-1}(\alpha_1 - 1) + 3^{n-2}(\alpha_2 - 1) + \ldots + (\alpha_n - 1) + 1$. In other

words, $\boldsymbol{\alpha}$ is an expression in base three of $j$. Thus, the elements of the adjacency matrix are given by

$$\begin{aligned}
&\Delta_{ij} = 1 && \text{if } i, j \text{ are nearest neighbours}\\
&\Delta_{ii} = -3 && \text{if } \boldsymbol{\alpha} \neq (\alpha_1^n)\\
&\Delta_{ii} = -2 && \text{if } \boldsymbol{\alpha} = (\alpha_1^n)\\
&\Delta_{ij} = 0 && \text{otherwise.}
\end{aligned}$$

The matrix $\boldsymbol{\Delta}$ has order $3^n$ and contains three diagonal blocks of identical structure of order $3^{n-1}$, which in turn are formed by three identical blocks of rank $3^{n-2}$ and so forth. Moreover, $\boldsymbol{\Delta}$ contains matrix elements which couple neighbouring blocks. The self-similar structure of the lattice is echoed by the self-similar structure of $\boldsymbol{\Delta}$.

The ternary labelling convention allows also for easily expressing the Cartesian coordinate of a cell. Let choose the barycentre of the initiator as origin of the coordinate axes, and indicate with $\boldsymbol{r}_i$ (with $i = 1, 2, 3$) the coordinates of the barycentres of the three cells of the first generation. The coordinate $\boldsymbol{r}$ of a cell identified by a sequence $\boldsymbol{\alpha}$ results from

$$\boldsymbol{r} = \sum_{k=1}^{n} \boldsymbol{r}_{\alpha_k} 2^{-k+1}.$$

An analogous labelling scheme can be used in higher-dimensional spaces by choosing a quaternary coding for $d = 3$ and so forth.

## 3. Evolution

We assume that the evolution of a non-conserved order parameter $\phi_i$ is governed by the following GL time-dependent equation:

$$\frac{\partial \phi_i}{\partial t} = -\Gamma \frac{\delta H[\{\phi_i\}]}{\delta \phi_i} + \xi_i(t) = -\Gamma \left[ -\sum_{j=1}^{N} \Delta_{ij}\phi_j + r + \frac{g}{N}\sum_{l=1}^{N} \phi_l^2(t) \right] \phi_i(t) + \xi_i(t). \tag{3}$$

Here $\xi_i(t)$ represents a Gaussian white noise with zero average

$$\langle \xi_i(t) \rangle = 0 \tag{4}$$

and variance

$$\langle \xi_i(t)\xi_j(t') \rangle = 2\Gamma T_f \delta_{ij} \delta(t - t') \tag{5}$$

where $T_f$ is the temperature of the final equilibrium state and the kinetic coefficient $\Gamma$ takes a constant value for non-conserved order parameter (NCOP) kinetics.

The popularity enjoyed by the spherical model in spite of its artificial nature is due to the fact that it displays a non-trivial behaviour while lending itself to an exact solution. Moreover, it enables one to study both the equilibrium and far-from equilibrium properties giving useful insight on more realistic models. As we show in this paper we are able to extend the model to a set of deterministic fractal lattices. In order to study the properties of the system one considers the equal time real space connected correlation function

$$C_{ij}(t) = \langle \phi_i(t)\phi_j(t) \rangle - \langle \phi_i(t) \rangle \langle \phi_j(t) \rangle.$$

At equilibrium, as we shall show below, the spherical model, with $r < 0$, on the SG of arbitrary embedding dimension does not phase separate at any non-zero temperature in contrast with standard Euclidean lattices of embedding dimension $d > 2$. It is well known

that on such regular lattices the model displays a low temperature ordered ('magnetic') phase with non-vanishing order parameter $\langle\phi\rangle$ and a high temperature disordered ('paramagnetic') phase. Due to the lack of translational symmetry on SG one cannot apply the Fourier analysis in order to decouple the different modes of the system. The translational invariance is therefore employed to diagonalize the non-local term $\Delta_{ij}\phi_j$ and to obtain equations for the Fourier components of the field $\phi_i$. On the other hand, in the present case the self-similar nature of the SG can be invoked allowing for the determination of the discrete spectrum and the degeneracy of the operator $\mathbf{\Delta}$. This route does not involve approximations and was exploited in the literature by some authors [5] and is alternative to the RG type of calculations.

In analogy with the Fourier transform one can construct a rotation operator in function space, starting from the basis functions constituted by the eigenvectors of $\mathbf{\Delta}$. For the sake of completeness we report hereafter the salient features of the spectrum associated with the operator $\mathbf{\Delta}$.

Due to the self-similar structure of the lattice the eigenvalues at the level $n$th are obtained from those at the $(n-1)$th level by means of an iterative procedure. The spectrum consists of three pieces: (a) a zero mode, associated with a uniform shift of the field $\{\phi_i\}$; (b) a subset of symmetric modes, and (c) a subset of antisymmetric modes [5]. The spectrum, relative to the $(n+1)$th stage lattice, contains all the eigenvalues of the $n$th lattice plus new eigenvalues generated through the map:

$$\lambda_n = (d+3-\lambda_{n+1})\lambda_{n+1}. \tag{6}$$

In practice one employs the inverse iteration, i.e. starts from the energy levels pertaining to small lattices and generates a denser and denser set of eigenvalues as the number of sites increases with increasing $n$.

$$\lambda_{n+1} = \frac{(d+3) \pm \sqrt{(d+3)^2 - 4\lambda_n}}{2}. \tag{7}$$

The null eigenvalue is associated with the eigenvector $(d+1)^{-n/2}(1, 1, 1, ..., 1)$.

The number of distinct eigenvalues is

$$\nu_n = 3 \times 2^{n-1} - 1. \tag{8}$$

In the case of $d = 2$ their degeneracy is given by $g_r = (3^{n-r-1}+3)/2$, where $r = 0, 1, .., n-1$ for the symmetric modes and $g_r = (3^{n-r-1}-1)/2$ with $r = 0, 1, .., n-2$ for the antisymmetric modes. For arbitrary embedding dimension $d$ the number of eigenvalues is given by the same formula (8), but the degeneracy becomes:

$$g_r = \frac{d-1}{2}\left[(d+1)^{n-r-1} + \frac{d+1}{d-1}\right] \tag{9}$$

for the symmetric states and

$$g_r = \frac{d-1}{2}[(d+1)^{n-r-1} - 1] \tag{10}$$

for the antisymmetric states. The density of states of the two-dimensional SG is shown in figure 2, whereas in figure 3 we display the integrated density of states, i.e. the fraction of states below a given energy.

After reordering the eigenvalues $\lambda$ in ascending order, we rename them $\epsilon_\alpha$ and count as distinct those which are degenerate. Notice that as $n \to \infty$ the smallest positive eigenvalue $\epsilon_1$ scales as $E_0/(d+3)^n$, where $E_0$ is an unimportant constant.

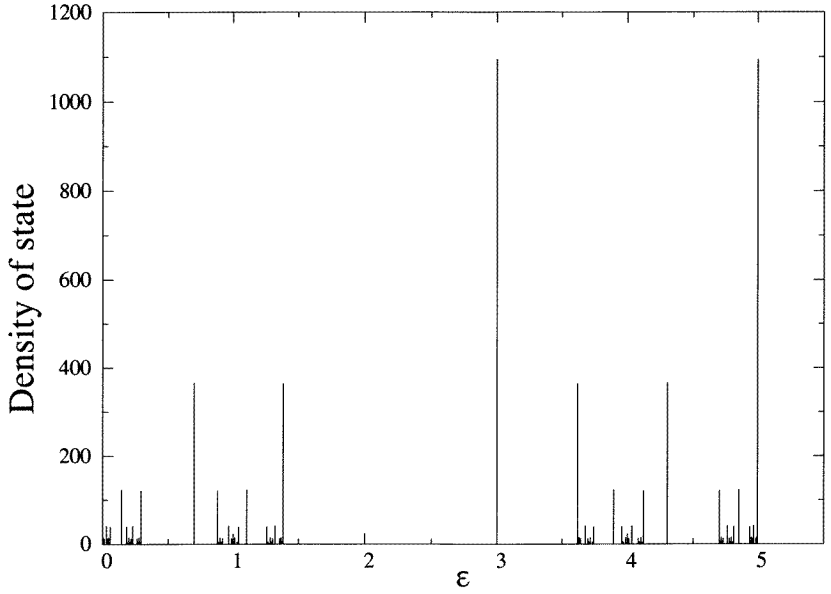


**Figure 2.** Density of states for the two-dimensional SG corresponding to nine generations. Notice the highly degenerate nature of the eigenfrequencies which form a double Cantor set.

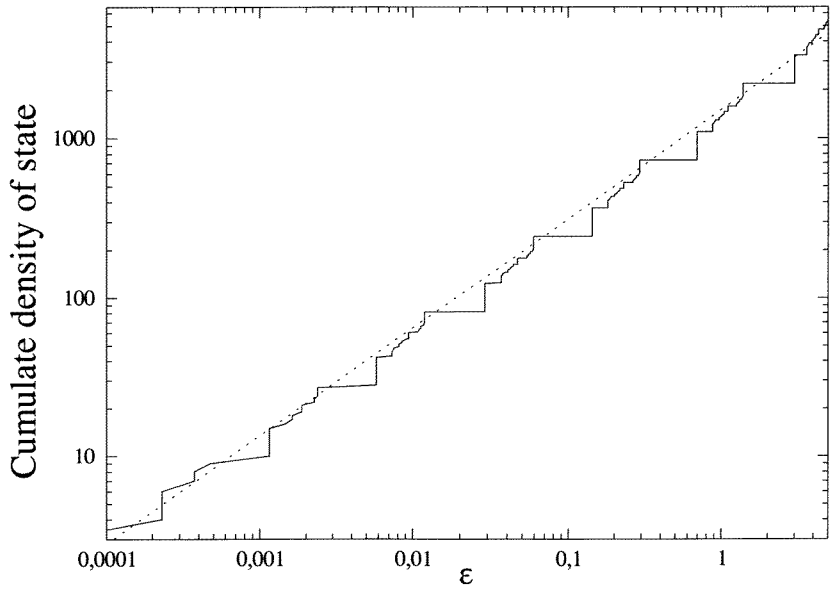


**Figure 3.** Integrated density of states versus smoothed density of states approximation for Sierpinski Gasket with $d = 2$. $N(\epsilon) \simeq \epsilon^{d_s/2}$ with $d_s = 2\ln(5)/\ln(3)$.

We, now, expand the field $\phi_i$ into a linear superposition of eigenvectors $\boldsymbol{u}^\alpha$ which are determined by the eigenvalue equation

$$-\sum_{j=1}^{N} \Delta_{ij} u_j^\alpha = \epsilon_\alpha u_i^\alpha \tag{11}$$

$$\phi_i(t) = \sum_{\alpha=0}^{N-1} \tilde{\phi}_\alpha(t) u_i^\alpha \tag{12}$$

where the $\tilde{\phi}_\alpha(t)$ are the projections of the field $\phi_i(t)$ on the basis $\boldsymbol{u}^\alpha$ and the greek indices denote the eigenvalues and latin indices the sites. Following [10] we consider a 'rotation' in the order parameter space which diagonalizes the interaction matrix, with $R_{\alpha i} = (u_i^\alpha)^*$.

$$\tilde{\phi}_\alpha(t) = \sum_{\alpha=0}^{N-1} R_{\alpha i} \phi_i(t). \tag{13}$$

Analogously the noise field $\xi_i(t)$ can be rotated without affecting its statistical properties:

$$\tilde{\xi}_\alpha(t) = \sum_{i=1}^{N} R_{\alpha i} \xi_i(t). \tag{14}$$

Taking into account the completeness of the eigenvectors, the norm of the order parameter is conserved

$$\frac{1}{N}\sum_{i=1}^{N} \phi_i^2(t) = \frac{1}{N}\sum_{\alpha=0}^{N-1} \tilde{\phi}_\alpha^2(t) \tag{15}$$

and equation (3) can be rewritten as

$$\frac{\partial}{\partial t}\tilde{\phi}_\alpha(t) = -\Gamma\left[\epsilon_\alpha + r + \frac{g}{N}\sum_{\beta=0}^{N-1} |\tilde{\phi}_\beta(t)|^2\right]\tilde{\phi}_\alpha(t) + \tilde{\xi}_\alpha(t). \tag{16}$$

In the large $N$-limit, summing over the index $\beta$ averages the system over an ensemble of configurations [11]. We shall denote the thermal averages by $\langle . \rangle$:

$$S(t) \equiv \frac{1}{N}\sum_{i=1}^{N}\langle \phi_i^2(t)\rangle = \frac{1}{N}\sum_{i=1}^{N}\phi_i^2(t) = \frac{1}{N}\sum_{\beta=0}^{N-1}|\tilde{\phi}_\beta(t)|^2 \tag{17}$$

where $S(t)$ must be determined self-consistently. To this purpose we introduce the auxiliary functions $Q(t)$ and $R(t)$, defined as

$$R(t) = r + gS(t) \tag{18}$$

and

$$Q(t) = \int_0^t \mathrm{d}t'\, R(t'). \tag{19}$$

One formally solves equation (16) by writing

$$\tilde{\phi}_\alpha(t) = \tilde{\phi}_\alpha(0) D_\alpha(t) + \int_0^t \mathrm{d}t' \frac{D_\alpha(t)}{D_\alpha(t')}\tilde{\xi}_\alpha(t') \tag{20}$$

where:

$$D_\alpha(t) = \exp(-\Gamma[\epsilon_\alpha t + Q(t)]). \tag{21}$$

Averaging over the noise field and taking into account the properties equations (4) and (5) we find:

$$\tilde{\phi}_\alpha(t) = \tilde{\phi}_\alpha(0) D_\alpha(t) \tag{22}$$

and

$$C_{\alpha\beta}(t) = \langle \tilde{\phi}_\alpha(t)\tilde{\phi}_\beta(t)\rangle. \tag{23}$$

Since $C_{\alpha\beta}(t) = C(\epsilon_\alpha, t)\delta_{\alpha\beta}$, one finds:

$$C_{\alpha\beta}(t) = \delta_{\alpha\beta} D_\alpha^2(t)\left[C_{\alpha\alpha}(0) + 2\Gamma T_f \int_0^t \mathrm{d}t' \frac{1}{D_\alpha^2(t')}\right]. \tag{24}$$

Thus, the equal time correlation function $C_{\alpha\alpha}(t)$ satisfies the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} C_{\alpha\alpha}(t) = -2\Gamma[\epsilon_\alpha + r + gS(t)]C_{\alpha\alpha}(t) + 2\Gamma T_f. \tag{25}$$

## 4. Static properties

In the limit $t \to \infty$ the left-hand side of equation (25) vanishes and we consider

$$\lim_{t\to\infty} C(\epsilon_\alpha, t) = \frac{T_f}{\epsilon_\alpha + R_\infty} \tag{26}$$

where $R_\infty = \lim_{t\to\infty}(r + gS(t))$. Within the limit $N \to \infty$ the vanishing of the quantity $R_\infty$ signals the appearance of the low temperature ordered phase below the critical temperature $T_c$. In order to handle carefully the limit $R_\infty \to 0$ we introduce the function

$$B(R_\infty) = \frac{1}{N}\sum_{\alpha=1}^{N-1} \frac{1}{\epsilon_\alpha + R_\infty} \tag{27}$$

and treat separately the first term in equation (27), so that the self-consistency condition (19) reads

$$R_\infty = r + \frac{g}{N}\frac{T_f}{R_\infty} + gT_f B(R_\infty). \tag{28}$$

We notice that since all the eigenvalues are non-negative the summand (27) is non-singular and is a monotonically decreasing function of $R_\infty$ within the interval $0 < R_\infty < \infty$. It is straightforward to verify that the nonlinear equation (28) always admits a solution for $R_\infty > 0$ and the second term in the right-hand side is irrelevant in the limit of large N. In the thermodynamic limit, i.e. when the number of generations $n \to \infty$, one should ask whether it is possible to observe a phase transition. We know from general arguments [3] that the answer is negative, but we shall illustrate how this behaviour comes about. If $R_\infty$ remains finite when $N \to \infty$, the second term in equation (28) vanishes. If we admit the possibility that $R_\infty \to 0$ as $1/N$ so that the quantity $M^2 = T_f/R_\infty N$, to be identified with the squared magnetization, is a finite positive constant, we can rewrite equation (28) as

$$0 = r + gM^2 + gT_f B(0). \tag{29}$$

But at $T_c$, which represents the critical temperature of the model, $M^2$ must vanish and we deduce

$$T_c = -\frac{r}{gB(0)} \tag{30}$$

i.e. a finite value of the critical temperature requires that the sum $B(0)$ does not diverge in the thermodynamic limit. By inspecting the behaviour of the function $B(0)$ as $N \to \infty$ one can assess the existence of a finite temperature phase transition. Numerically, as shown in figure 4, we find that the sum diverges with the linear dimension $L = 2^n$ of the lattice as $B(0) \sim L^\alpha$ where $\alpha = (d_w - d_f)$, where $d_w = \ln(d+3)/\ln 2$ is the fractal dimension of the walk on the SG (i.e. the mean square displacement $x^2(t)$ scales with time as $t^{2/d_w}$). In $d = 2$, $\alpha = 0.739$, while in $d = 3$ $\alpha = 0.585$ [12]. In figure 4 we also show results for $B(0)$ for different values of the embedding dimension, $d$. Notice that since the exponent $\alpha$

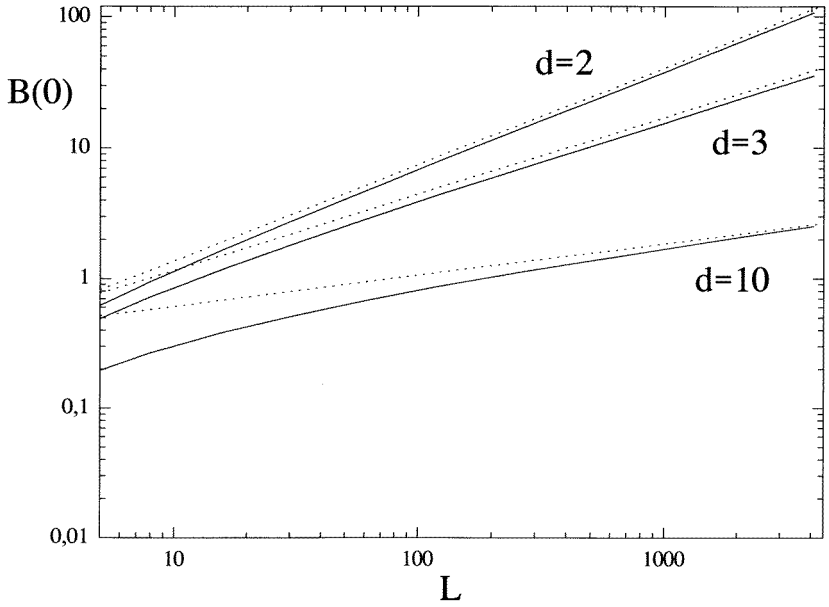


**Figure 4.** Double logarithmic plot of the function $B(0)$ versus the linear size $L$ for different values of the embedding dimension $d$ ($d = 2$, $d = 3$ and $d = 10$). The broken lines represent the behaviour $L^{d_w - d_f}$ for each case.

converges very slowly to $0^+$, when $d \to \infty$ the system never displays a phase transition to a low temperature phase for any finite $T_c$. We recall the situation on Euclidean lattices where $d_w$ is always equal to 2, so that $\alpha = 0$ in two dimensions (logarithmic behaviour) and there is no divergence in three dimensions.

The leading behaviour of $B(0)$ can be captured by means of a simple scaling argument. One assumes a smoothed density of states $\rho(\epsilon) \sim \epsilon^{d_s/2-1}$, where $d_s = 2d_f/d_w$ [17]. In the thermodynamic limit we can approximate the sum by an integral

$$B(0) \sim \int_{\epsilon_{\min}} \frac{\rho(\epsilon)}{\epsilon} \mathrm{d}\epsilon \sim \epsilon_{\min}^{d_s/2-1} \qquad \text{if } d_s < 2.$$

Since the smallest non-zero eigenvalue $\epsilon_{\min}$, for very large $n$, scales with the size of the gasket as $E_0/(d+3)^n$, we can rewrite it as $\epsilon_{\min} \sim E_0 2^{n d_w}$ and conclude that $B(0) \sim 2^{n(d_w - d_f)} = L^{(dw - d_f)}$.

Such a result can also be obtained analytically. To this end we reconsider the structure of the sum $B(0)$; as we have already remarked, for large lattices the smallest eigenvalues are well approximated by the formula $\epsilon_s = E_0/(d+3)^{n-s+1}$, and their degeneracy is proportional to $\sim (d+1)^s$. We use this approximation in order to compute the largest contribution to the sum $B(0)$.

$$B(0) \simeq \frac{(d+3)^n}{(d+1)^n} \sum_{s=1}^{\nu_n} \frac{(d+1)^s}{(d+3)^s} \tag{31}$$

for values of the index $n \to \infty$ the sum converges

$$B(0) \simeq \frac{(d+3)^n}{(d+1)^n} \sum_{s=1}^{\nu_n} \frac{(d+1)^s}{(d+3)^s} \sim \text{constant } 2^{(d_w - d_f)n} \tag{32}$$

since the last sum represents a convergent geometric series. This allows us to regard the approximation involved in replacing the discrete sum $B(0)$ by an integral weighted with the smoothed density of states $\rho(\epsilon) \sim \epsilon^{d_s/2-1}$, as able to capture the main features of the model. Similar approximations have been used to calculate specific heats of fractal structures and proved to be effective. In figure 4 we display results obtained by using $\rho(\epsilon)$ against those obtained by using the exact spectrum. In spite of the very fragmented structure of the real density of states, the method yields good results also for the dynamical properties as we shall see below. To conclude the present section we can rule out the possibility of a phase transition at low temperatures, in agreement with general arguments *à la Landau*, since SG are always below the lower critical dimensionality. On the other hand, as we shall see later one can observe a phase transition in a self-affine lattice formed by an infinite stack of parallel two-dimensional SG connected along the normal direction to form a regular one-dimensional lattice. These systems in fact have an infinite ramification order.

## 5. Dynamical properties

We shall consider, now, the approach to equilibrium when the system is quenched from an uncorrelated high temperature initial condition realized by equally populating all modes, i.e. by choosing $C_{\alpha\beta}(t = 0) = C_0\delta_{\alpha\beta}$ for any value of $\alpha$. The parameters are the same as before. Since it is well known [2] that the ordering process is controlled by the zero temperature fixed point we shall neglect the noise term. In the case $T_f = 0$ the leading asymptotic behaviour of $C_{\alpha\alpha}(t)$ can be obtained by a simple matching method. One starts from observing that $S(t)$ must approach a constant value for large times ($t \to \infty$)

$$\lim_{t\to\infty} S(t) = -\frac{r}{g}. \tag{33}$$

Therefore from the evolution equation for $C_{\alpha\alpha}(t)$ and the self-consistency condition one writes

$$S(t) = \mathrm{e}^{-2\Gamma Q(t)}\frac{C_0}{N}\sum_{\alpha}\mathrm{e}^{-2\Gamma\epsilon_\alpha t} + 2\Gamma T_f\frac{1}{N}\sum_{\alpha}\int_0^t \mathrm{d}\tau\ \mathrm{e}^{-2\Gamma[\epsilon_\alpha(t-\tau)-(Q(t)-Q(\tau))]}. \tag{34}$$

Setting $T_f = 0$ and going to the continuum limit by the substitution $\frac{1}{N}\sum_\alpha \to \int \rho(\epsilon)\,\mathrm{d}\epsilon$, which proved to give the correct results in the previous section,

$$S(t) = \mathrm{e}^{-2\Gamma Q(t)}\int \epsilon^{d_s/2-1}\mathrm{e}^{-2\Gamma\epsilon t}\,\mathrm{d}\epsilon. \tag{35}$$

The leading behaviour of expression (35) is $\sim t^{-d_s/2}\mathrm{e}^{-2\Gamma Q(t)}$. Imposing the matching condition equation (33) we find

$$2\Gamma Q(t) \sim -\frac{d_s}{2}\ln t. \tag{36}$$

We conclude that the correlator $C(\epsilon_\alpha, t)$ has the following scaling behaviour

$$t^{d_s/2}\mathrm{e}^{-2\Gamma\epsilon_\alpha t}. \tag{37}$$

Since the smallest eigenvalues, as noticed before, are well approximated by the relation

$$\epsilon_\alpha = \left(\frac{1}{d+3}\right)^{n-\alpha} E_0 \tag{38}$$

we can rewrite

$$\epsilon_\alpha = E_0\left(\frac{2^\alpha}{2^n}\right)^{d_w} \tag{39}$$

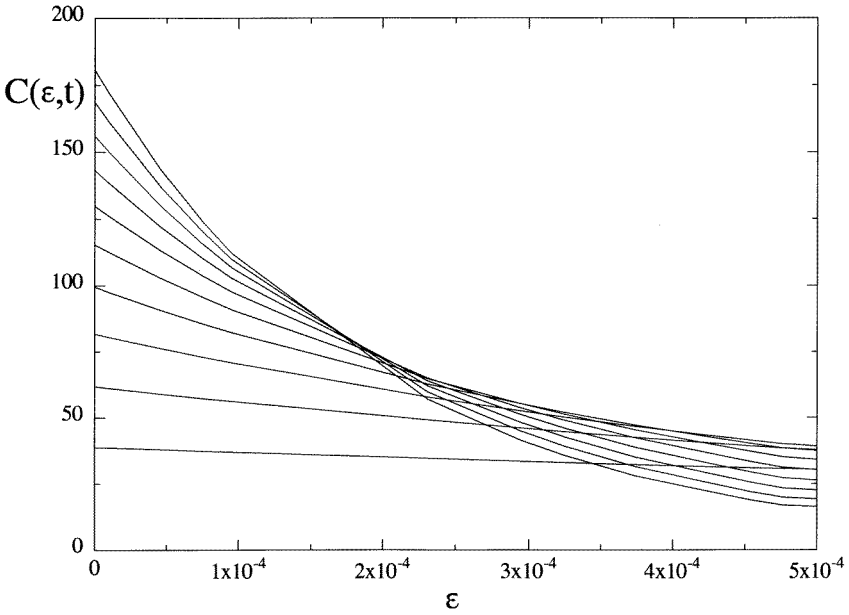


**Figure 5.** Plot of the structure function $C(\epsilon, t)$ versus $\epsilon$ for different times.

and defining $m = 2^{\alpha}$ and $q = m/L$ the exponential function can be expressed as

$$\exp(-\epsilon_{\alpha} t) = \exp(-E_0 q^{d_w} t). \tag{40}$$

Thus, we obtain the scaling function

$$C(\epsilon_{\alpha}, t) \sim t^{d_s/2} \exp(-E_0 q^{d_w} t). \tag{41}$$

Such a relation should be compared with the scaling function pertaining to standard lattices, where $d_s \to d$ and $d_w \to 2$, and the structure factor displays the scaling behaviour $t^{d/2} \mathrm{e}^{-q^2 t}$.

We remark that the quantity $q$ plays the role of the inverse of the wavelength, in agreement with the Alexander–Orbach definition [17] of a characteristic length on fractals. As we have already observed during the study of the equilibrium properties of the system there is no genuine phase transition at finite temperatures due to the existence of large amplitude low-energy modes, the analogue on these lattices of long wavelength fluctuations, which destroy the long range order (Goldstone modes) on Euclidean lattices. The evolution of the structure function $C(\epsilon, t)$ at different times is displayed in figure 5, while in figure 6 we show its maximum value as a function of time. Another quantity of interest, since it gives a measure of the size of the domains, is the average

$$R^2(t) = \frac{\sum_{ij} |i - j|^2 \langle \phi_i(t) \phi_j(t) \rangle}{\sum_{ij} \langle \phi_i(t) \phi_j(t) \rangle} \tag{42}$$

which we observe numerically to evolve in time as $t^{2/d_w}$, where $d_w$ is the random walk exponent on the SG lattice (see figure 7). This behaviour is expected since in NCOP dynamics the domain walls move in a random walk fashion.

Finally, we considered the data collapse which can be obtained by rescaling the structure functions and the time scales as illustrated in figure 8.

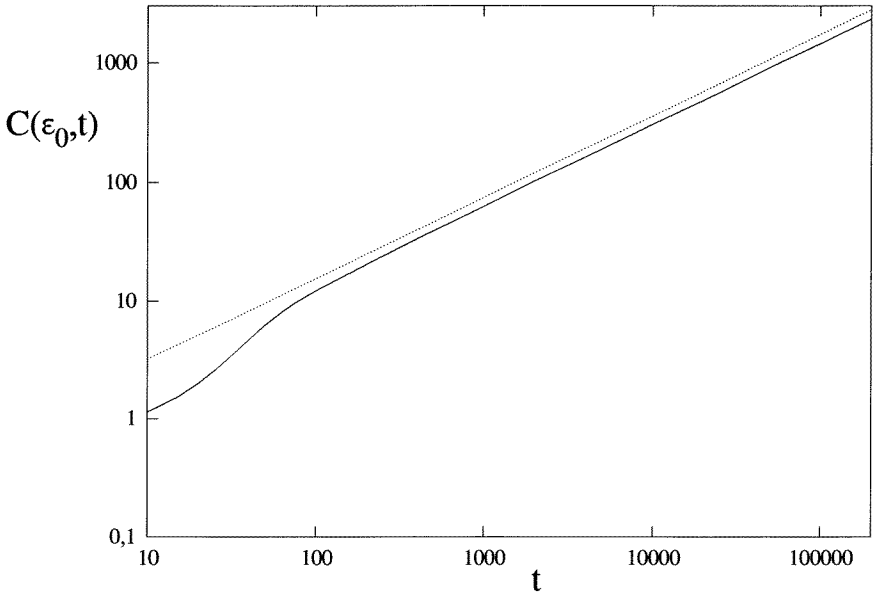


**Figure 6.** Maximum value of the structure function $C(\epsilon, t)$ for the NCOP dynamics on the $d = 2$ Sierpinski Gasket versus $t$. The broken line represents the curve $t^{d_s/2}$.

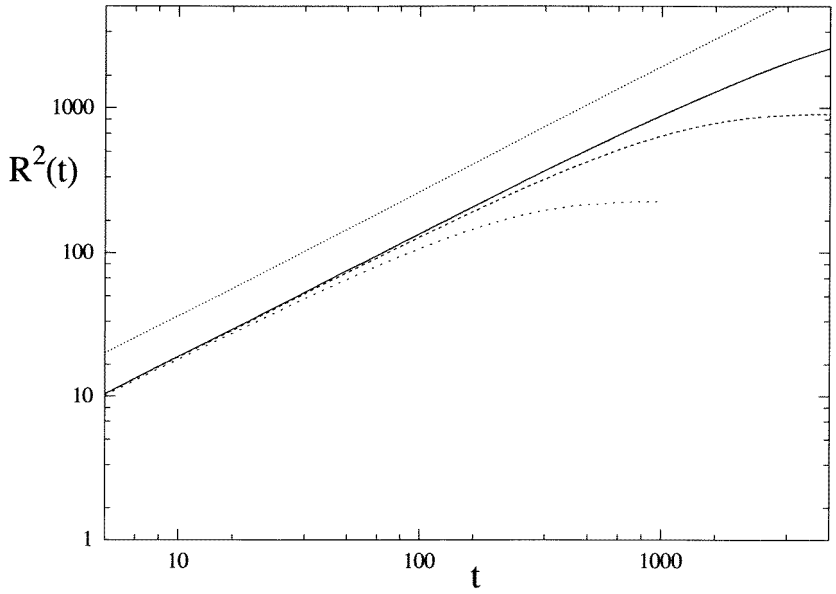


**Figure 7.** The typical domain-size $R^2(t)$ as a function of time and curve $t^{2/d_w}$ (broken curve).

## 6. Characteristic relaxation times and finite temperature effects

In order to study the finite temperature properties we derive an integro-differential equation by introducing the function $\zeta(t) = \exp(2\Gamma Q(t))$ which is related to the $\alpha = 0$ component

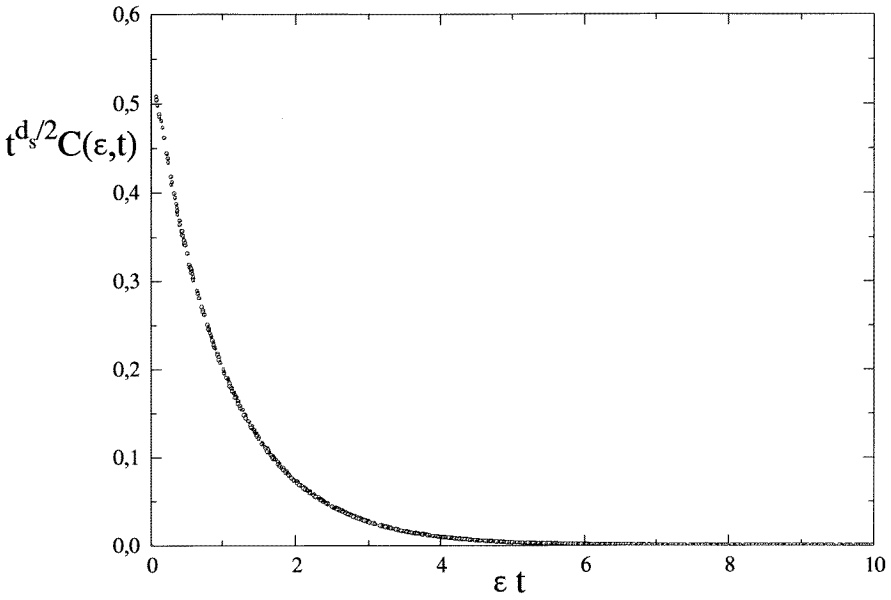


**Figure 8.** Data collapse for the structure function $C(\epsilon, t)$ for the NCOP dynamics on the $d = 2$ Sierpinski lattice, obtained by rescaling $C(\epsilon, t)$ by $t^{-d_s/2}$ and the time scale by $\epsilon_\alpha^{-1}$.

of $C_{\alpha\alpha}(t)$ via the relation

$$\zeta(t) = \frac{C_{00}(0)}{C_{00}(t)}. \tag{43}$$

Using equation (25) one obtains

$$\frac{\mathrm{d}\zeta(t)}{\mathrm{d}t} = 2\Gamma r\zeta(t) + 2\Gamma g\frac{1}{N}\sum_\alpha \mathrm{e}^{-2\Gamma\epsilon_\alpha t}C_{\alpha\alpha}(0) + 2\Gamma g T_f\frac{1}{N}\sum_\alpha \mathrm{e}^{-2\Gamma\epsilon_\alpha(t-\tau)}\zeta(\tau). \tag{44}$$

The integro-differential equation can be solved by Laplace transformation methods by introducing the function $\hat{\zeta}(s) = \int_0^\infty \zeta(t)\mathrm{e}^{-st}\,\mathrm{d}t$

$$\hat{\zeta}(s) = \frac{1 + 2\Gamma g\Delta J(s)}{s - 2\Gamma r - 2\Gamma T_f g J(s)} \tag{45}$$

where the quantity $J(s)$ is given by the expression

$$J(s) = \frac{1}{N}\sum_\alpha \int_0^\infty \mathrm{e}^{-2\Gamma\epsilon_\alpha t}\mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{N}\sum_\alpha \frac{1}{2\Gamma\epsilon_\alpha + s}. \tag{46}$$

By approximating

$$J(s) = \frac{1}{N}\sum_\alpha \frac{1}{2\Gamma\epsilon_\alpha + s} \sim s^{d_s/2-1}\int \mathrm{d}x\frac{x^{d_s/2-1}}{1+x} = s^{d_s/2-1}w_{d_s} \tag{47}$$

we have introduced $w_{d_s}$ a dimension dependent constant. The long time behaviour of interest is determined by the nature of the singularities of the function $\hat{\zeta}(s)$ in the complex plane. When $r < 0$ equation (45) has a singularity approximately at

$$s \sim \left(\frac{-T_f g w_{d_s}}{r}\right)^{1/(1-d_s/2)}. \tag{48}$$

The deeper the quench, i.e. $T_f$ small, the longer the typical relaxation time $\tau$

$$\tau = \left(\frac{-r}{gT_f}\right)^{1/(1-d_s/2)}. \tag{49}$$

Instead, if $r > 0$, the magnetization decays quickly to zero in a time interval of order $\tau(2\Gamma|r|)^{-1}$.

We outline an alternative method to derive the scaling of the relaxation time as a function of the temperature quench $T_f$. Let us start by considering the equation for the height of the peak of the equilibrium structure factor as a function of the temperature. In the spherical case, the equilibrium structure function $C_{\rm eq}(\epsilon)$ does not diverge at any finite temperature $T_f$ and its expression is

$$\lim_{\epsilon\to 0} C_{\rm eq}(\epsilon) = \frac{T_f}{R_\infty} \tag{50}$$

since $R_\infty$ vanishes as $T_f \to 0$, we can assume $R_\infty = cT_f^\alpha$ with $\alpha$ a positive exponent. For determining $R_\infty$ we use

$$R_\infty = r + gT_f \int \frac{\rho(\epsilon)}{\epsilon + R_\infty}\,{\rm d}\epsilon \sim r + gT_f K_{d_s} R_\infty^{d_s/2-1} \tag{51}$$

where $K_{d_s}$ is a numerical constant. As $T_f \to 0$ we find

$$R_\infty \sim \left(-\frac{g}{r}T_f\right)^{1/(1-d_s/2)} \tag{52}$$

thus

$$C_{\rm eq}(\epsilon = 0) = \left(-\frac{r}{g}\right)^{1/(1-d_s/2)} T_f^{1-1/(1-d_s/2)}. \tag{53}$$

In order to extract the typical relaxation time we recall that in a quench at $T_f = 0$ the peak of the dynamical structure factor grows in time as $C(0,t) \sim -(r/g)t^{d_s/2}$ (see equation (41)). If one assumes that this growth persists in a system at finite temperature up to a characteristic time $\tau$ and then settles to the equilibrium value given by equation (54) we find from $C(0,t) \sim C_{\rm eq}(0)$:

$$\tau \sim \left(-\frac{r}{g}T_f\right)^{-1/(1-d_s/2)} \tag{54}$$

consistent with the value obtained from equation (49).

## 7. Renormalization group treatment

The dynamical scaling obeyed by the structure factor in the case of SGs is a clear signature of the existence of an underlying invariance of the GL equation under RG transformations. Coniglio and Zannetti have constructed explicitly the dynamical RG in the case of Euclidean lattices [11]. In this section we shall extend their method to the present case. To this end one considers two stages:

(1) Elimination of hard modes, i.e. of modes with energy $E_c/\lambda < \epsilon_\alpha < E_c$ where $\lambda > 1$, where $E_c$ is some UV cut-off.

(2) Rescaling of energy, time, and order parameter in order to render the coarse grained equations equivalent to the original one.

As noticed by Bray [2] within the spherical model the hard modes can be ignored because in the asymptotic regime they reach equilibration very fast. The rescalings involved in the RG transformation are

$$\epsilon' = \lambda^x \epsilon \tag{55}$$

$$t' = \lambda^{-z} t \tag{56}$$

$$\tilde{\phi}'(\epsilon', t') = \lambda^{-y} \tilde{\phi}(\epsilon, t). \tag{57}$$

Since it is convenient to consider the equation of evolution for the structure function $C(\epsilon, t)$ we introduce the exponent $\alpha$

$$C'(\epsilon', t') = \lambda^{\alpha} C(\epsilon, t). \tag{58}$$

From the definition

$$C(\epsilon_1, t) = \langle \tilde{\phi}(\epsilon_1, t) \tilde{\phi}(\epsilon_2, t) \rangle \delta(\epsilon_1 - \epsilon_2) \tag{59}$$

on sees that $\alpha$ is not an independent exponent but is related to $y$ by the relation

$$\alpha = d_f - 2y. \tag{60}$$

By inserting equations (55)–(58) the equation of motion can be rewritten as

$$\frac{\partial C'(\epsilon', t')}{\partial t'} = -2\Gamma[\epsilon' \lambda^{z-x} + \lambda^z R(\lambda^z t')] C'(\epsilon', t') + 2\Gamma T_f \lambda^{z+\alpha}. \tag{61}$$

Comparing equation (58) with (64) one finds the recursion relations

$$T_f' = \lambda^{x+\alpha} T_f \tag{62}$$

$$\Gamma' = \lambda^{z-x} \Gamma \tag{63}$$

$$R'(t') = \lambda^x R(t' \lambda^z) \tag{64}$$

which have the fixed point solution $x = z$, $T_f = 0$ and $\lim_{t \to \infty} R(t) = 0$. The exponent $x$ turns out to be equal to $d_w$ as shown in section 4 (as one can see from halving a lattice of linear size $L$). In order to determine the exponent $\alpha$, we consider the self-consistency condition

$$\int \mathrm{d}\epsilon' \rho(\epsilon') C(\epsilon', t) = -\frac{r}{g} \tag{65}$$

which explicitly reads

$$\int \mathrm{d}\epsilon' \, \rho(\epsilon') C(\epsilon', t) \sim \lambda^{\alpha + x d_s/2} \int \mathrm{d}\epsilon \, \epsilon^{d_s/2 - 1} C(\epsilon, t). \tag{66}$$

We conclude that $x d_s/2 + \alpha = 0$, i.e. $\alpha = -d_f$ and $y = d_f$. Therefore if we choose $\lambda = t^{1/z} = t^{1/d_w}$ we obtain from equation (58) the following form of the structure function

$$C(\epsilon, t) = t^{d_f/d_w} f(\epsilon t) \tag{67}$$

where $f(x)$ is a scaling function, which is consistent with the results of the previous section.

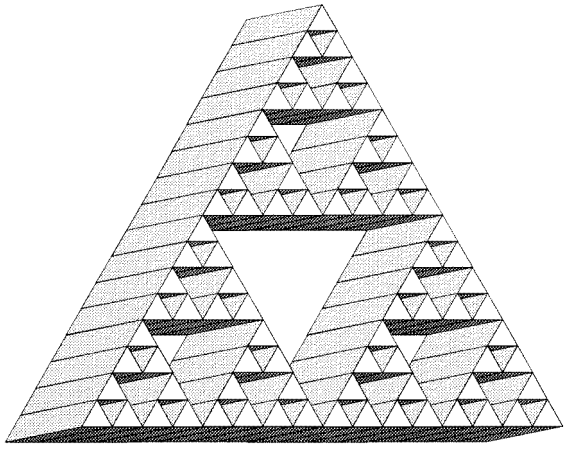

**Figure 9.** Toblerone self-affine lattice obtained by direct product of two-dimensional SG with linear chains.

## 8. Self-affine lattices

The reason for the absence of a finite temperature phase transition on SG is associated with the existence of arbitrarily large clusters of sites, which can be removed from the rest of the structure by cutting only a finite number of interactions. When such a feature is eliminated we can observe a true low temperature phase. In order to achieve this goal we construct a self-affine structure characterized by an embedding dimension $d = 3$ and by an infinite ramification order. We consider the direct product of a planar SG with a linear chain. This kind of anisotropic lattice has been termed Toblerone [14]. It can be built by considering a sequence of $N_z$ parallel SGs normal to the $z$-direction. Each site in a given plane is connected not only to the nearest-neighbour sites within the same plane, but also to the corresponding sites belonging to the two nearest planes (see figure 9). The Toblerone fractal dimension is $d_f = d_f^{SG} + 1$, where $d_f^{SG}$ is the fractal dimension of the SG. By following the same method employed in section 4 we consider the behaviour of the function $B(R_\infty)$ which can be written as

$$B(R_\infty) = \frac{1}{N_z N} \sum_{\alpha=1}^{N-1} \sum_{m=1}^{N_z} \frac{1}{\epsilon_\alpha + 2(1-\cos q) + R_\infty} \tag{68}$$

where $q = (2\pi m)/N_z$ and $m = 1, 2 \ldots, N_z$. Taking the limit $N_z \to \infty$ and $N \to \infty$ and setting $R_\infty = 0$ we can convert the sum over $q$ into an integral by means of the substitution:

$$\frac{1}{N_z} \sum_{m=1}^{N_z} \to \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}$$

rewriting equation (68) we find

$$B(R_\infty) = \frac{1}{N} \sum_{\alpha=1}^{N-1} \int_{-\pi}^{\pi} \frac{\mathrm{d}q}{2\pi} \frac{1}{\epsilon_\alpha + 2(1-\cos q) + R_\infty} \tag{69}$$

$$B(R_\infty) = \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{\tau}{(\tau^2 - 1)^{1/2}} \tag{70}$$

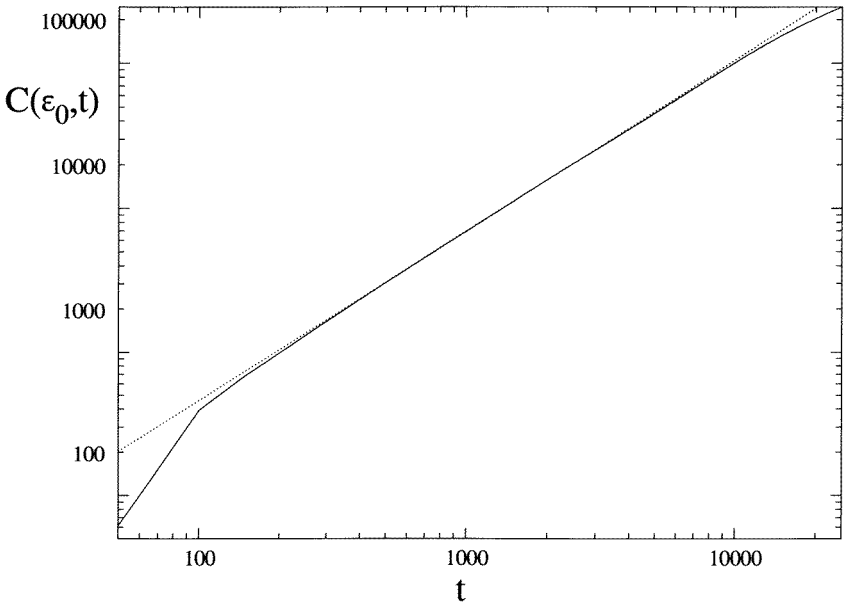


**Figure 10.** Maximum value of $C(\epsilon, t)$ as a function of time against theoretical prediction for the Toblerone lattice $t^{d_s^{SG}/2+1/2}$.

where $\tau = \frac{\epsilon_\alpha + R + 2}{2}$. One can easily see that the sum in equation (70) converges to a finite value in this case. The density of state argument yields the following results in the limit $R_\infty \to 0$.

$$B(0) \sim \int_{\epsilon_{\min}} \frac{2+\epsilon}{\sqrt{\epsilon^2+4\epsilon}} \epsilon^{d_s^{SG}/2-1} \, \mathrm{d}\epsilon \tag{71}$$

where $d_s^{SG} = 2\ln(d+1)/\ln(d+3)$ as above. In conclusion the critical temperature is finite and the system is always above its lower critical dimensionality. The critical temperature is readily computed from the formula

$$T_c = -\frac{r}{gB(0)}. \tag{72}$$

The dynamical scaling exponent $z$ can be extracted with the help of the same method as in section 5. The self-consistency condition for $S(t)$ together with the assumption that it approaches a finite constant as $t \to \infty$ leads to the estimate

$$2\Gamma Q(t) = -\frac{d_s^{SG}+1}{2} \ln(t). \tag{73}$$

Correspondingly the correlation function displays the following scaling behaviour

$$C(\epsilon_\alpha, k, t) \sim t^{(d_s^{SG}+1)/2} \exp(-q^{d_w^{SG}} t) \exp(-k^2 t). \tag{74}$$

The above predictions have been checked numerically. We found $T_c$ to be nearly $-0.61 r/g$. In figure 10 we display the behaviour of the peak of the structure function as a function of time.

## 9. Conclusions

Besides the mathematical interest for these models one can think of these as representing the process of aggregation of particles induced by the presence of a fractal substrate. In this case, the system does not form compact structures during the growth, but the mass of the aggregate scales with the average radius as $M \sim r^{d_f}$. This process has been modelled in the literature introducing an anomalous dependence of the diffusion constant on the mass of the aggregate. Our proposal instead mimics the void regions which form during the growth process by means of a fractal background lattice.

In the present we have exactly solved the spherical model on a class of deterministic fractals, the SG and Toblerone lattices by exact diagonalization of the adjacency matrix. The solution illustrates explicitly the general predictions based on graph theory that structures with finite order of ramification do not display finite temperature phase transition [15, 16]. Our results also indicate that the spectral dimension is a fairly good parameter for computing integral quantities, in spite of the very singular structure of the real density of states. In addition we were able to derive all the off-equilibrium properties of the system and obtain explicitly the scaling functions. We believe that our results can offer a useful check for testing approximations concerning dynamical and statical properties of GL theories on self-similar structures. The present results can be extended to include a conservation law of the order parameter and to discuss new approximate treatments of scalar GL theories.

## Appendix. Equilibrium properties

In the present appendix we shall outline the calculations of the equilibrium properties of the model. On the lattice the partition function associated with the Hamiltonian

$$Z_N[\{h_i\}] = \Pi_{i=1}^{N} \int_{-\infty}^{\infty} \mathrm{d}\phi_i \, \mathrm{e}^{-\beta H[\phi_i] + \beta \sum_i h_i \phi_i}. \tag{75}$$

Where we have included an external field $h_i$ coupled linearly to $\phi_i$ and $\beta = (k_B T_f)^{-1}$. In order to separate the macroscopic component $P$ of the field we employ the following identity

$$1 = N \int_{-\infty}^{\infty} \mathrm{d}P^2 \, \delta(N P^2 - \sum_i \phi_i^2) \tag{76}$$

and rewrite $Z_N$ as

$$\begin{aligned} Z_N[\{h_i\}] &= N \int_{-\infty}^{\infty} \mathrm{d}P^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\phi_i \, \Pi_{i=1}^{N} \mathrm{e}^{-\beta H[\phi_i] + \beta \sum_i h_i \phi_i + \mathrm{i}\lambda(N P^2 - \sum_i \phi_i^2)} \\ Z_N[\{h_i\}] &= N \int_{-\infty}^{\infty} \mathrm{d}P^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\phi_i \, \exp\left[-\beta N \left(\frac{r}{2} P^2 + \frac{g}{4} P^4 - \mathrm{i}\frac{\lambda}{\beta} P^2\right)\right] \\ &\quad \times \int_{-\infty}^{\infty} \Pi_{i=1}^{N} \mathrm{e}^{-\beta/2[-\sum_{ij} \phi_i \Delta_{ij} \phi_j + 2\mathrm{i}\lambda/\beta \sum_i \phi_i^2]} \mathrm{e}^{\beta \sum_i h_i \phi_i}. \end{aligned} \tag{77}$$

In the case of a uniform external field ($h_i = h$), eliminating the $\phi_i$ fields, $Z_N$ reads:

$$\begin{aligned} Z_N[\{h_i\}] = N \mathrm{e}^{\frac{N}{2} \ln(2\pi/\beta)} &\int_{-\infty}^{\infty} \mathrm{d}P^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \exp\left(-\beta N \left[\frac{r}{2} P^2 + \frac{g}{4} P^4 - \mathrm{i}\frac{\lambda}{\beta} P^2\right]\right) \\ &\mathrm{e}^{-1/2 \sum_\alpha \ln(\epsilon_\alpha + 2\mathrm{i}\lambda/\beta)} \exp\left[-\mathrm{i} N \frac{\beta^2 h^2}{4\lambda}\right]. \end{aligned}$$

$Z_N$ can be evaluated by saddle point estimate in the limit $N \to \infty$ imposing the conditions

$$\frac{\partial Z}{\partial \lambda} = 0 \tag{78}$$

$$\frac{\partial Z}{\partial P^2} = 0 \tag{79}$$

$$\frac{2\mathrm{i}\lambda}{\beta} = r + gP^2 \tag{80}$$

$$P^2 = \frac{1}{\beta N}\sum_{\alpha}\frac{1}{\epsilon_\alpha + 2\mathrm{i}\lambda/\beta} - \frac{\beta^2 h^2}{4\lambda^2}. \tag{81}$$

Eliminating $\lambda$ with the help of equations (80) and (81) we find

$$P^2 = \frac{1}{\beta N}\sum_{\alpha}\frac{1}{\epsilon_\alpha + r + gP^2} + \frac{h^2}{(r+gP^2)^2}. \tag{82}$$

The last term equals $M^2$, the square of the average magnetization $M = \frac{1}{N}\sum_i \langle\phi_i\rangle = \frac{1}{\beta N}\,\mathrm{d}\ln Z_N/\mathrm{d}h$. By using equation (81) we find explicitly

$$M = \frac{\beta}{2\mathrm{i}\lambda} = \frac{h}{r + gP^2}. \tag{83}$$

The existence of a spontaneous magnetic phase implies that in zero magnetic external field $M \neq 0$, i.e. the following condition must be fulfilled

$$\lim_{h\to 0}\,[r + gP^2] = 0. \tag{84}$$

The equation of state reads

$$\left[r + gM^2 + \frac{g}{N}T_f\sum_{\alpha}\frac{1}{\epsilon_\alpha + r + gS_\infty + gM^2}\right]M = h \tag{85}$$

where $S_\infty$ is given by

$$S_\infty = \frac{g}{N}T_f\sum_{\alpha}\frac{1}{\epsilon_\alpha + r + gS_\infty + gM^2}. \tag{86}$$

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